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Three-dimensional AdS gravity and extremal CFTs at $c = 8m$

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Abstract: We note that Witten's proposed duality between extremal $c = 24k$ CFTs and three-dimensional anti-de Sitter gravity may possibly be extended to central charges that are multiples of 8, for which extremal self-dual CFTs are known to exist up to $c = 40$. All CFTs of this type with central charges $c \geqslant 24$, provided that they exist, have the required mass gap and may serve as candidate duals to three-dimensional gravity at the corresponding values of the cosmological constant. Here, we compute the genus one partition function of these theories up to $c = 88$, we give exact and approximate formulas for the degeneracies of states, and we determine the genus two partition functions of the theories up to $c = 40$.

Keywords: Field Theories in Lower Dimensions, AdS-CFT Correspondence.

Contents

1. Introduction

Three-dimensional gravity [1 − 3] is a promising candidate for a potentially soluble [4, 5] gravitational theory with non-trivial structure. In this respect, the most interesting case is that of negative cosmological constant, where the AdS/CFT correspondence maps the problem of solving the theory to the more concrete one of identifying the dual CFT, and where the theory admits, in addition to the standard conical solutions [6, 7], the BTZ black hole solutions [8, 9]. The latter share many properties with their four-dimensional counterparts and, in particular, have a nonzero entropy which should be accounted for by the microstates of any candidate dual CFT. These considerations led Witten [10] to re-examine the theory and to propose a duality with a certain class of self-dual CFTs. Below, we outline the salient points of the proposal.

An important property of three-dimensional gravity is that it admits a description in terms of a Chern-Simons theory [11]. For AdS_3 gravity with cosmological constant $\Lambda = -1/\ell^2$, plus a possible gravitational Chern-Simons term, the corresponding action,

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R + \frac{2}{\ell^2}\right) + \frac{k'}{4\pi}\int \mathrm{Tr}\left(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right), \tag{1.1}$$

can be recast [5] into the following combination of Chern-Simons actions

$$\begin{aligned} S &= S_L - S_R\,, \\ S_{L,R} &= \frac{k_{L,R}}{4\pi}\int \mathrm{Tr}\left(A_{L,R}\wedge dA_{L,R} + \frac{2}{3}A_{L,R}\wedge A_{L,R}\wedge A_{L,R}\right). \end{aligned} \tag{1.2}$$

Here, $A^a_{L,R} = \omega^a \mp \frac{1}{\ell}e^a$ are the $\mathrm{SO}(2,1)\times\mathrm{SO}(2,1)$ gauge fields, expressed in terms of the dreibein e^a and the dual spin connection $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$, while $k_{L,R} = \frac{\ell}{16G} \pm \frac{k'}{2}$. For pure AdS gravity, where $k' = 0$ and $k_L = k_R = k = \ell/16G$, it was first shown by Brown and Henneaux [12] that the Poisson-bracket algebra of the asymptotic symmetries consists of two copies of the Virasoro algebra with common central charge $c = 3\ell/2G = 24k$; in modern terminology this is understood as the central charge of the dual CFT. For $k' \neq 0$, $c_{L,R} = 24k_{L,R}$ are different from each other.

Regarding the quantum theory, the Chern-Simons formulation has the conceptual advantages that it makes finiteness of three-dimensional gravity manifest and provides a good starting point for formulating perturbation theory. However, turning to nonperturbative aspects, there are a few problems, the most important of which being that the Chern-Simons theory appears to have too few degrees of freedom to account for the degeneracy of BTZ black holes (unless one associates them with boundary excitations [13]). As the above make clear that this formulation must be used with caution, the approach advocated by Witten in [10] was to use the Chern-Simons description only as a guide for obtaining the relevant values of the central charges of the boundary CFT, and then to try determining the latter by imposing modular invariance and the existence of a mass gap.

The relevant values of $c_{L,R}$ are obtained from the quantization conditions on the Chern-Simons couplings $k_{L,R}$. To find these conditions, it must be specified whether the gauge group is precisely $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$ or a cover thereof. Specifically, for an n-fold diagonal cover of $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$, the quantization conditions are given by [10]

$$k_L \in \frac{1}{n}\mathbb{Z}\ (n\ \text{odd})\,,\ \frac{1}{2n}\mathbb{Z}\ (n\ \text{even})\,, \qquad k' = k_L - k_R \in \mathbb{Z}\ \left(\text{or}\ \frac{1}{3}\mathbb{Z}\right)\,, \tag{1.3}$$

where the relaxed condition $k' \in \frac{1}{3}\mathbb{Z}$ arises when ω is treated like a connection on the tangent bundle rather than like an ordinary gauge field. In the simplest $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$ case, $k_{L,R}$ are integers and $c_{L,R} = 24k_{L,R}$ are multiples of 24; for pure gravity this means that $k = \ell/16G$ is an integer. We stress that there is no a priori reason for picking a particular value of n; the only restriction is that we cannot take $n \to \infty$, as this would allow us to continuously vary $c = 3\ell/2G$ in contradiction with Zamolodchikov's c-theorem.

The cases $c = 24k$, $k \in \mathbb{Z}$, are rather special in that they are the only ones that allow for the possibility of holomorphic factorization. In the self-dual case, where the space of states of the theory consists only of the vacuum representation, holomorphic factorization implies that the partition function factorizes as

$$\mathcal{Z}_c(\tau,\bar\tau) = Z_c(\tau)\bar{Z}_c(\bar\tau)\,, \tag{1.4}$$

with $Z_c(\tau)$ and $\bar{Z}_c(\bar\tau)$ being separately modular invariant. Although there exists no compelling argument for considering only the cases $c = 24k$, the decomposition (1.2) of the action as a sum of two terms is suggestive of a factorization of the form (1.4), and the absence of CFTs with the required properties at lower central charges points towards the value $c = 24$ above which there is a proliferation of holomorphic CFTs. Be that as it may, the assumption of holomorphic factorization tremendously simplifies things, as one may uniquely determine the partition functions of the sought CFTs by imposing modular invariance and requiring that the primaries associated with black-hole states enter at the highest possible level. For $c = 24$ and $c = 48$, the (holomorphic) partition functions read

$$\begin{aligned} Z_{24}(\tau) &= j(\tau) - 744 \\ &= q^{-1} + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + 20245856256\,q^4 + \ldots\,, \end{aligned} \tag{1.5}$$

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and

$$\begin{aligned} Z_{48}(\tau) &= j^2(\tau) - 1488\, j(\tau) + 159769 \\ &= q^{-2} + 1 + 42987520\, q + 40491909396\, q^2 + 8504046600192\, q^3 + \dots , \end{aligned} \tag{1.6}$$

where $j(\tau)$ is the modular j-function and $q = e^{2\pi i\tau}$. The partition function in (1.5) defines a very special theory among the 71 holomorphic CFTs believed to exist at $c = 24$ [14]. It was first constructed by Frenkel, Lepowsky and Meurman [15] (see also [16]) by considering 24 chiral bosons on the Leech lattice and using a $\mathbb{Z}_2$ orbifold to project out the 24 dimension-1 primaries. The 196884 dimension-2 operators correspond to one Virasoro descendant plus 196883 primaries whose number is the dimension of the lowest non-trivial representation of the largest sporadic group, the monster group. In fact, each coefficient in (1.5) equals the number of descendants at this level plus the dimension of an irreducible representation of the monster; this observation forms part of monstrous moonshine, an unexpected connection between modular functions and finite simple groups. For the partition function (1.6) and, in general, all $Z_{24k}(\tau)$ with $k \geqslant 2$, the corresponding CFTs have not been identified, but the number of available lattices in these dimensions makes their existence plausible. Furthermore, the counting of microstates in the CFTs under consideration yields, for all values of k, an entropy that is very close to the corresponding Bekenstein-Hawking entropy of the BTZ black hole. These facts led Witten to propose that three-dimensional quantum gravity with $\ell/16G = k \in \mathbb{Z}$ is dual to the $c = 24k$ series of extremal CFTs. In particular, for the most negative possible value of the cosmological constant, the dual CFT has been conjectured to be the $c = 24$ monster theory.

2. Genus one extremal partition functions for $c = 8m$

The main purpose of this paper is to note that the arguments mentioned above may apply, with minor modifications, to the case where the central charge is a multiple of 8, $c = 8m$ with $m \in \mathbb{Z}$. In the Chern-Simons formulation this corresponds to taking a three-fold diagonal cover of $SO(2,1) \times SO(2,1)$ as the gauge group. Note that the resulting values of the Chern-Simons couplings, $k_{L,R} \in \frac{1}{3}\mathbb{Z}$, fit nicely with the last quantization condition in (1.3). In the cases $c = 8m$, holomorphic factorization is no longer possible in the strict sense, but one can still have holomorphic factorization up to a phase. This, along with the requirement that the primaries associated with black holes appear at the right level, uniquely specifies the partition function for each value of m. In what follows, we describe the construction of these partition functions, and we state exact and approximate formulas for the corresponding degeneracies of states.

2.1 Partition functions

The starting point of our construction is the well-known fact that, for a CFT of central charge $c = 8m$, there exists the possibility that the partition function factorizes as in (1.4), but with the holomorphic part picking up a phase under $T : \tau \to \tau + 1$,

$$Z_{8m}(\tau) \to e^{-2\pi i m/3} Z_{8m}(\tau) \ , \tag{2.1}$$

and with the antiholomorphic part picking up the opposite phase so that the full partition function is modular invariant (see e.g. [14, 17]). Assuming that this is the case and furthermore assuming as in [10] that the theory is self-dual, the construction proceeds as follows. In the absence of primary fields, the partition function of such a theory would be just the vacuum Virasoro character,

$$Z_{0,8m}(\tau) = q^{-m/3} \prod_{n=2}^{\infty} \frac{1}{1-q^n} = q^{-m/3} \sum_{n=0}^{\infty} \left(P(n) - P(n-1)\right) q^n \,, \tag{2.2}$$

where $P(n)$ denotes the number of partitions of n. This partition function clearly cannot account for the degeneracy of the BTZ black holes and, in addition, transforms non-trivially under $S : \tau \to -1/\tau$. To remedy these problems we would like to add primaries, to be identified with operators creating BTZ black holes, in such a way that modular invariance up to a phase is restored. To figure out the conformal weight of such states, we note that, choosing the additive constant in L_0 so that its eigenvalue is $h - \frac{c}{24}$ where h is the conformal weight, L_0 is related to the mass and angular momentum of a BTZ black hole according to

$$L_0 = \frac{1}{2}(\ell M + J) \,, \tag{2.3}$$

and the Bekenstein-Hawking entropy reads [18]

$$S_{BH}(m, L_0) = 4\pi \sqrt{\frac{c}{24} L_0} = 4\pi \sqrt{\frac{m}{3} L_0} \,, \tag{2.4}$$

with similar relations for the antiholomorphic sector. The minimal mass of a black hole corresponds to the case $\ell M = |J|$, i.e. $L_0 = 0$, for which the entropy vanishes. Therefore, the primaries associated with the black-hole states should appear for $L_0 > 0$, i.e. for $h \geqslant h_m$ where

$$h_m \equiv \left[\frac{m}{3}\right] + 1 \,. \tag{2.5}$$

On the other hand, according to a result of Höhn [19], the dimension of the lowest primary in a self-dual CFT has an upper limit given by $h \leqslant h_m$. Therefore, our requirements can be satisfied only if $h = h_m$, that is, if the full partition function has the form

$$Z_{8m}(\tau) = q^{-m/3} \left(\prod_{n=2}^{\infty} \frac{1}{1-q^n} + \mathcal{O}(q^{[m/3]+1}) \right) \,. \tag{2.6}$$

Such partition functions are called extremal and have the remarkable property that they are uniquely determined once one imposes modular invariance up to a phase. Namely, the requirement (2.1) fixes a self-dual partition function with $c = 8m$ to be a weighted polynomial of weight $m/3$ generated by $j^{1/3}(\tau)$, with the general form [19]

$$Z_{8m}(\tau) = j^{m/3}(\tau) \sum_{r=0}^{[m/3]} a_r j^{-r}(\tau) \,. \tag{2.7}$$

The coefficients a_r are then determined by matching the terms of order $q^{r-m/3}$, $r = 0, \ldots, [m/3]$, with those in (2.6) as in [10] (see also [20]). The results of this analysis are given below.

For $c = 8$, 16, we have $h_m = 1$ meaning that the extra states enter at level one above the vacuum. Therefore there exists no mass gap and the extra states must correspond to massless fields arising as a result of a gauge symmetry. In fact, as the self-dual partition functions for $c = 8$ and $c = 16$ are well-known and believed to be unique, the result can be immediately anticipated. Indeed, applying the matching procedure we find

$$Z_8(\tau) = j^{1/3}(\tau) = q^{-1/3} + 248\, q^{2/3} + 4124\, q^{5/3} + 34752\, q^{8/3} + 213126\, q^{11/3} + \dots\,, \quad (2.8)$$

which is the vacuum character of the level 1 affine $\hat{E}_8$ theory (or $q^{-1/3}$ times the McKay-Thompson series of class 3C for the monster) and

$$\begin{aligned} Z_{16}(\tau) &= j^{2/3}(\tau) \\ &= q^{-2/3} + 496\, q^{1/3} + 69752\, q^{4/3} + 2115008\, q^{7/3} + 34670620\, q^{10/3} + \dots \end{aligned} \quad (2.9)$$

which is the vacuum character of the level 1 affine $\hat{E}_8 \times \hat{E}_8$ theory. Due to the presence of Kac-Moody symmetries, these extremal CFTs cannot be directly relevant to pure AdS_3 gravity but, possibly, to extensions of AdS_3 gravity including gauge fields with Chern-Simons interactions [10].

For $c = 24$, 32, 40, we have $h_m = 2$, i.e. the extra states enter at level two above the vacuum and the required mass gap exists. For $c = 24$ we find the partition function (1.5) discussed earlier on. For $c = 32$ and $c = 40$, we obtain the partition functions

$$\begin{aligned} Z_{32}(\tau) &= j^{4/3}(\tau) - 992\; j^{1/3}(\tau) \\ &= q^{-4/3} + 139504\, q^{2/3} + 69332992\, q^{5/3} + 6998296696\, q^{8/3} \\ &\quad + 330022830080\, q^{11/3} + \dots\,, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} Z_{40}(\tau) &= j^{5/3}(\tau) - 1240\, j^{2/3}(\tau) \\ &= q^{-5/3} + 20620\, q^{1/3} + 86666240\, q^{4/3} + 24243884350\, q^{7/3} \\ &\quad + 2347780456448\, q^{10/3} + \dots\,. \end{aligned} \quad (2.11)$$

These partition functions have been first obtained by Höhn in [19] and the corresponding CFTs have been identified with $\mathbb{Z}_2$ orbifolds of theories defined on even unimodular lattices of the respective rank possessing no vectors of squared length 2. Proceeding in this manner, we may in principle specify the partition function for any value of m. Explicit formulas up to $c = 88$ (omitting the cases $c = 48$, 72 already considered in [10]) are given below.

$$\begin{aligned}
Z_{56}(\tau) &= j^{7/3}(\tau) - 1736\, j^{4/3}(\tau) + 401661\, j^{1/3}(\tau) \\
&= q^{-7/3} + q^{-1/3} + 7402776\, q^{2/3} + 33941442214\, q^{5/3} \\
&\qquad + 16987600857280\, q^{8/3} + 2998621352249926\, q^{11/3} \dots , \\
Z_{64}(\tau) &= j^{8/3}(\tau) - 1984\, j^{5/3}(\tau) + 705057\, j^{2/3}(\tau) \\
&= q^{-8/3} + q^{-2/3} + 278512\, q^{1/3} + 13996663144\, q^{4/3} + 19414403055040\, q^{7/3} \\
&\qquad + 769385603725340\, q^{10/3} + 1062805058989221728\, q^{13/3} + \dots , \\
Z_{80}(\tau) &= j^{10/3}(\tau) - 2480\, j^{7/3}(\tau) + 1496361\, j^{4/3}(\tau) - 132423391\, j^{1/3}(\tau) \\
&= q^{-10/3} + q^{-4/3} + q^{-1/3} + 173492852\, q^{2/3} + 4695630250012\, q^{5/3} \\
&\qquad + 8461738959649848\, q^{8/3} + 4293890043969667206\, q^{11/3} + \dots , \\
Z_{88}(\tau) &= j^{11/3}(\tau) - 2728\, j^{8/3}(\tau) + 1984269\, j^{5/3}(\tau) - 302198519\, j^{2/3}(\tau) \\
&= q^{-11/3} + q^{-5/3} + q^{-2/3} + 2365502\, q^{1/3} + 907649518712\, q^{4/3} \\
&\qquad + 4712143513485758\, q^{7/3} + 4723281033156413468\, q^{10/3} + \dots .
\end{aligned} \tag{2.12}$$

These partition functions have not been, to our knowledge, previously identified in the literature. It would be interesting to examine whether corresponding CFTs actually exist.

2.2 Microstate counting

In what follows, we will verify that the partition functions constructed above can account for the degeneracy of the BTZ black hole states. According to Witten's interpretation, the new states appearing at each level are divided into primary states, corresponding to black holes, and Virasoro descendants of lower-lying primary states, corresponding to lower-mass black holes dressed with boundary excitations. Therefore, the number of microstates associated with black holes of a given mass is given by the number of primaries at the corresponding level. The total number of states $D(m, L_0)$ at a given eigenvalue $L_0 = h - \frac{m}{3}$ is read off from the relation

$$Z_{8m}(\tau) = \sum_{L_0 + m/3 = 0}^{\infty} D(m, L_0) q^{L_0} , \tag{2.13}$$

and the number $d(m, L_0)$ of primaries is then obtained by subtracting the number of descendants at this level. Once $d(m, L_0)$ is determined, we can define the microscopic entropy

$$S(m, L_0) = \ln d(m, L_0) \ . \tag{2.14}$$

In practice, the contribution of descendant states to $D(m, L_0)$ is negligible and we can trade $d(m, L_0)$ for $D(m, L_0)$. In any case, the entropy computed by means of (2.14) turns out to be quite close to the semiclassical entropy (2.4), as explicitly shown on table 1 for $m = 3, \dots, 8$ and for the first few values of L_0.

However, this agreement is a mild check of the proposed duality since it is mostly controlled by modular invariance[1] rather than by the detailed structure of the theory. To

[1] We thank Edward Witten for stressing this point.

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m	L_0	d	S	S_{BH}
3	1	196883	12.1904	12.5664
	2	21296876	16.8741	17.7715
	3	842609326	20.5520	21.7656
4	2/3	139503	11.8458	11.8477
	5/3	69193488	18.0524	18.7328
	8/3	6928824200	22.6589	23.6954
5	1/3	20619	9.9340	9.3664
	4/3	86645620	18.2773	18.7328
	7/3	24157197490	23.9078	24.7812

m	L_0	d	S	S_{BH}
6	1	42987519	17.5764	17.7715
	2	40448921875	24.4233	25.1327
	3	8463511703277	29.7668	30.7812
7	2/3	7402775	15.8174	15.6730
	5/3	33934039437	24.2477	24.7812
	8/3	16953652012291	30.4615	31.3460
8	1/3	278511	12.5372	11.8477
	4/3	13996384631	23.3621	23.6954
	7/3	19400406113385	30.5963	31.3460

Table 1: Degeneracies, microscopic entropies and semiclassical entropies for the first few values of m and L_0.

see this, we recall that the Petersson-Rademacher formula [21, 22] completely determines the coefficients $F(l)$ of $q^{l-c/24}$ in the expansion of a modular form of weight w in terms of the corresponding polar coefficients $F(n)$, $n-\frac{c}{24}<0$, according to [23]

$$\begin{aligned} F(l) \,=\, 2\pi \sum_{n-c/24<0} \left(\frac{\frac{c}{24}-n}{l-\frac{c}{24}}\right)^{(1-w)/2} F(n) \\ \times \sum_{k=1}^{\infty} \frac{1}{k}\,\mathrm{Kl}\left(l-\frac{c}{24}, n-\frac{c}{24}; k\right) I_{1-w}\left(\frac{4\pi}{k}\sqrt{\left(\frac{c}{24}-n\right)\left(l-\frac{c}{24}\right)}\right) \,, \end{aligned} \tag{2.15}$$

where $\mathrm{Kl}(a,b\,;k)$ is the Kloosterman sum

$$\mathrm{Kl}(a,b\,;k) \equiv \sum_{d\in(\mathbb{Z}/k\mathbb{Z})^*} \exp\left(\frac{2\pi \mathrm{i}}{k}\left(d\,a+d^{-1}\,b\right)\right) \,, \tag{2.16}$$

and $I_\nu(z)$ is a modified Bessel function of the first kind. Using this expression for the coefficients $D(m,L_0)$ of the partition function $Z_{8m}(\tau)$ and noting that for an extremal CFT the polar coefficients are the same as those in the vacuum character (2.2), namely $D\left(m,n-\frac{m}{3}\right)=P(n)-P(n-1)$ for $n=0,\ldots,[m/3]$, we find

$$\begin{aligned} D(m,L_0) \,=\, 2\pi \sum_{n=0}^{[m/3]} \sqrt{\frac{\frac{m}{3}-n}{L_0}}\,(P(n)-P(n-1)) \\ \times \sum_{k=1}^{\infty} \frac{1}{k}\,\mathrm{Kl}\left(L_0, n-\frac{m}{3}; k\right) I_1\left(\frac{4\pi}{k}\sqrt{\left(\frac{m}{3}-n\right)L_0}\right) \,, \end{aligned} \tag{2.17}$$

where we note that the $n=1$ term vanishes since $P(1)=P(0)$. In this expression, the only factor that depends on the details of the theory is $P(n)-P(n-1)$.

Eq. (2.17) is an exact result, which can be used to derive various approximate expressions, appropriate for limiting cases. A first simplification is to use Weil's estimate $\mathrm{Kl}(a,b\,;k)\simeq\sqrt{k}$ to obtain the expression

$$D(m,L_0) \simeq 2\pi \sum_{n=0}^{[m/3]} \sqrt{\frac{\frac{m}{3}-n}{L_0}}\,(P(n)-P(n-1)) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} I_1\left(\frac{4\pi}{k}\sqrt{\left(\frac{m}{3}-n\right)L_0}\right) \,, \tag{2.18}$$

which turns out to be in excellent agreement with the actual number of microstates. The semiclassical results usually quoted in the literature are obtained by taking the large-m and large-L_0 limit, using the asymptotics $I_1(z) \simeq e^z/\sqrt{2\pi z}$, and keeping only the $n = 0$ and $k = 1$ terms in the two summations. Doing so, all information about the details of the theory (apart from the ground-state degeneracy) disappears, and one obtains the Hardy-Ramanujan formula

$$D(m, L_0) \simeq \frac{1}{\sqrt{2}} \frac{(m/3)^{1/4}}{L_0^{3/4}} \exp\left(4\pi\sqrt{\frac{m}{3}L_0}\right) , \tag{2.19}$$

which is valid for any CFT of central charge $c = 8m$ and leads to the entropy

$$S(m, L_0) \simeq S_{BH}(m, L_0) - \frac{3}{2}\ln S_{BH}(m, L_0) + \ln\frac{m}{3} + \ln\frac{4\sqrt{2}}{\pi} , \tag{2.20}$$

which is the Bekenstein-Hawking result plus the logarithmic corrections [24, 25]. Therefore, in the semiclassical limit one recovers the Bekenstein-Hawking entropy, as guaranteed by Cardy's formula [26], while the qualitative agreement for smaller values of m and L_0, as those shown on table 1, is due to the convergence properties of the sums in (2.18).

On the other hand, the exact formula (2.17) (or its approximation (2.18)) allows for a controlled expansion that makes it possible to determine the various corrections to the semiclassical results. In particular we note that the partition numbers $P(n)$, being the coefficients of $q^{n-1/24}$ in $\eta^{-1}(q)$, admit themselves the Petersson-Rademacher expansion

$$P(n) = 2\pi\left(\frac{\frac{1}{24}}{n - \frac{1}{24}}\right)^{3/4} \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{Kl}\left(n - \frac{1}{24}, -\frac{1}{24}; k\right) I_{3/2}\left(\frac{4\pi}{k}\sqrt{\frac{1}{24}\left(n - \frac{1}{24}\right)}\right) . \tag{2.21}$$

Substituting (2.21) (or a suitable approximation thereof) in (2.18), and expanding around the semiclassical limit ($m, L_0 \to \infty$, L_0/m fixed), one may in principle calculate all leading and subleading corrections to the Bekenstein-Hawking formula in a systematic manner. It has been reported in [10] that a study of these corrections is in progress.

3. Genus two extremal partition functions for $c = 8m$

The above construction of extremal partition functions can be extended to the genus two case. In this case, the modular group is $PSp(4, \mathbb{Z})$ and the partition function is expressed as a combination of Siegel modular forms (see e.g. [27, 28]) with the appropriate transformation properties. To determine the various coefficients, there are two alternative methods. The first method [29] is to consider the limit in which the genus two surface degenerates to two tori joined at a point[2] and to require that the partition function factorize into a sum of products involving two torus one-point functions and a certain power of the pinching parameter; this method is quite straightforward but for large c it requires detailed information about the behavior of Siegel forms under degeneration, which might be hard to

[2] Alternatively, one can consider the degeneration to a single torus with two points joined. Consistency requires that the two approaches be equivalent.

obtain. The second method, suggested in [10] and used in [30] for calculating the genus two partition functions for $c = 24$, 48, 72, is to determine the partition function from the singularities of a six-point function of twist fields, possibly using some restricted information from the factorization condition; this method seems to be more involved than the first, but is actually easier to apply for large values of c. In what follows, we will use the first method to compute the genus two partition functions for the extremal CFTs with $c = 8m$ up to $c = 40$.

Before we do so, let us summarize the relevant formalism. In the genus two case, the moduli space is the quotient of the Siegel upper half space by the modular group $PSp(4,\mathbb{Z})$, and is parameterized by the period matrix Ω which transforms according to $\Omega \to (A\Omega + B)(C\Omega + D)^{-1}$. A Siegel modular form F_w of weight w is defined as a holomorphic function of Ω that transforms as follows

$$F_w(\Omega) \to (\det(C\Omega + D))^w \, F_w(\Omega) \, . \tag{3.1}$$

Any such form admits a Laurent expansion in the parameters $q = e^{2\pi i \Omega_{11}}$, $r = e^{2\pi i \Omega_{12}}$ and $s = e^{2\pi i \Omega_{22}}$ or, equivalently, in q, s and $u = r + r^{-1} - 2$. The ring of Siegel modular forms is generated [31] by four forms of weight 4, 6, 10 and 12, namely the two Eisenstein series ψ_4 and ψ_6 and the two cusp forms χ_{10} and χ_{12}. The former two admit the following expansions

$$\begin{aligned}
\psi_4(\Omega) &= \frac{1}{4} E_4(q)E_4(s) + 3600\, qsu + 60\, qsu^2 + \mathcal{O}(q^2, s^2) \, , \\
\psi_6(\Omega) &= \frac{1}{16} E_6(q)E_6(s) + 2646\, qsu + \frac{63}{2} qsu^2 + \mathcal{O}(q^2, s^2) \, ,
\end{aligned} \tag{3.2}$$

where $E_4(q) = 1 + 240\, q + \mathcal{O}(q^2)$ and $E_6(q) = 1 - 504\, q + \mathcal{O}(q^2)$ are the standard Eisenstein series, while the latter two can be expanded in a similar manner with the leading terms given in terms of the modular discriminant $\Delta(q)$ by $u\Delta(q)\Delta(s)$ and $96\Delta(q)\Delta(s)$, respectively. To discuss the degeneration limit, we will use the formalism of [29, 32]. The genus two surface can be constructed according to a "sewing" procedure [33], where two tori with modular parameters $q_{1,2} = e^{2\pi i \tau_{1,2}}$ are joined by excising a disc of radius $|\epsilon|$ from each torus (ϵ being is a complex "pinching" parameter) and making an appropriate identification of two annular regions around the excised discs. The degeneration limit corresponds to $\epsilon \to 0$, and the relations between the parameters q, r, u and q_1, q_2, ϵ are as follows

$$\begin{aligned}
q &= q_1 \left(1 - \frac{1}{12}\epsilon^2 E_2(q_2)\right) + \mathcal{O}(\epsilon^4) \, , \\
s &= q_2 \left(1 - \frac{1}{12}\epsilon^2 E_2(q_1)\right) + \mathcal{O}(\epsilon^4) \, , \\
u &= \epsilon^2 + \frac{1}{2}\epsilon^4 \left(1 + \frac{1}{36} E_2(q_1)E_2(q_2)\right) + \mathcal{O}(\epsilon^6) \, ,
\end{aligned} \tag{3.3}$$

where $E_2(q) = 1 - 24\, q + \mathcal{O}(q^2)$. The behavior of the Siegel modular forms under degeneration is determined by substituting eq. (3.3) into the corresponding expansions. Doing so for $\psi_{4,6}$ and applying the Ramanujan identities, we find

$$\begin{aligned}\psi_4 &= \frac{1}{4}E_{4,1}E_{4,2} + \epsilon^2\left(3600\, q_1 q_2 + \frac{1}{144}(E_{6,1}F_{6,2} + F_{6,1}E_{6,2} - 2F_{6,1}F_{6,2})\right) + \mathcal{O}(\epsilon^4)\,,\\ \psi_6 &= \frac{1}{16}E_{6,1}E_{6,2} + \epsilon^2\left(2646\, q_1 q_2 + \frac{1}{384}(E_{4,1}^2F_{8,2} + F_{8,1}E_{4,2}^2 - 2F_{8,1}F_{8,2})\right) + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{3.4}$$

where $F_6 \equiv E_2E_4$, $F_8 \equiv E_2E_6$, while $E_{4,i} \equiv E_4(q_i)$ and so on. Expanding the ϵ^2 term up to order q_i^2, we obtain

$$\begin{aligned}\psi_4 = \frac{1}{4}E_{4,1}E_{4,2} - 5\,\epsilon^2\Big(&q_1 + q_2 + \big(18(q_1^2+q_2^2) - 288\, q_1q_2\big) + 216\, q_1q_2(q_1+q_2)\\ &-132192\, q_1^2q_2^2 + \mathcal{O}(q_i^3)\Big) + \mathcal{O}(\epsilon^4)\,,\\ \psi_6 = \frac{1}{16}E_{6,1}E_{6,2} + \frac{21}{8}\epsilon^2\Big(&q_1 + q_2 + \big(66(q_1^2+q_2^2) - 48\, q_1q_2\big) - 39456\, q_1q_2(q_1+q_2)\\ &-608256\, q_1^2q_2^2 + \mathcal{O}(q_i^3)\Big) + \mathcal{O}(\epsilon^4)\;.\end{aligned} \tag{3.5}$$

Likewise, for $\chi_{10,12}$ we may use the formulas of [29, 32][3] to find

$$\begin{aligned}\chi_{10} &= \epsilon^2\left(\Delta_1\Delta_2 - \frac{1}{12}\epsilon^2\left(q_1q_2 - 48\, q_1q_2(q_1+q_2) + 2304\, q_1^2q_2^2 + \mathcal{O}(q_i^3)\right) + \mathcal{O}(\epsilon^4)\right)\,,\\ \chi_{12} &= 96\Delta_1\Delta_2 - 8\,\epsilon^2\left(q_1q_2 + \frac{9225}{22}q_1q_2(q_1+q_2) + \frac{1252764}{11}q_1^2q_2^2 + \mathcal{O}(q_i^3)\right) + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{3.6}$$

where again $\Delta_i \equiv \Delta(q_i)$.

Returning to the problem at hand, the general form of the genus two partition functions for the extremal CFTs under consideration is dictated by $PSp(4,\mathbb{Z})$ modular invariance. Namely, the genus two partition function $Z^{(2)}_{8m}(\Omega)$ for $c = 8m$ should be invariant up to a phase under the translations $T_1 : \Omega_{11} \to \Omega_{11}+1$, $T_2 : \Omega_{22} \to \Omega_{22}+1$, $U : \Omega_{12} \to \Omega_{12}+1$ and the reflection $V : \epsilon \to -\epsilon$, and should satisfy the transformation law $Z^{(2)}_{8m}(\Omega) \to (-\tau_{1,2})^{2m/3}Z^{(2)}_{8m}(\Omega)$ under $S_{1,2} : \tau_{1,2} \to -1/\tau_{1,2}\,, \epsilon \to -\epsilon/\tau_{1,2}$ [29]. These conditions lead us to expect that this function has the form

$$Z^{(2)}_{8m}(\Omega) = \chi_{10}^{-m/3}(\Omega)\sum_{r=0}^{[2m/5]}\chi_{10}^r(\Omega)\Psi_{4m-10r}(\Omega)\,, \tag{3.7}$$

where $\Psi_{4m-10r}(\Omega)$ is an entire $PSp(4,\mathbb{Z})$ modular form of the indicated weight, constructed out of ψ_4, ψ_6 and χ_{12}. At the degeneration limit, this must factorize as follows

$$Z^{(2)}_{8m}(\Omega) = \sum_h \epsilon^{2h-2m/3}\sum_{\phi_i,\phi_j\in\mathcal{H}_h}\mathcal{G}^{\phi_i\phi_j}Z^{(1)}_{8m}(\phi_i,\tau_1)Z^{(1)}_{8m}(\phi_j,\tau_2)\,, \tag{3.8}$$

where the sums run over all levels h and over all operators at each corresponding subspace $\mathcal{H}_h$ of the Hilbert space, $Z^{(1)}_{8m}(\phi_i,\tau)$ denotes the torus one-point function of the operator ϕ_i (normalized so that $Z^{(1)}_{8m}(1,\tau)$ equals the torus partition function $Z^{(1)}_{8m}(\tau)$), and $\mathcal{G}^{\phi_i\phi_j}$

[3]The weight-12 form F_{12} appearing in [29] is given by $F_{12} = \frac{1}{9}(704\,\psi_4^3 - 512\,\psi_6^2 + 76032\,\chi_{12})$.

is the inverse Zamolodchikov metric (normalized so that $\mathcal{G}^{11} = 1$). By inserting (3.5) and (3.6) in (3.7) and expanding in ϵ, one may then determine each Ψ_{4m-10r} by comparing the $\mathcal{O}(\epsilon^{2r-2m/3})$ terms with the corresponding terms in (3.8). For $r = 0$ this matching only requires knowledge of the leading terms in (3.5) and (3.6) and of the torus partition functions, while for each $r > 0$ this matching requires knowledge of the $\mathcal{O}(\epsilon^{2r})$ terms in (3.5) and (3.6) and of the torus one-point functions of all operators at level r. Furthermore, recalling that in the extremal CFTs of interest the primaries appear at levels $h \geqslant [m/3]+1$, and noting that the torus one-point function of a primary with $h \geqslant 11$ vanishes [30], we see that for $m \leqslant 32$ we may determine all Ψ_{4m-10r} by knowing just the one-point functions of the Virasoro descendants which are easy to obtain using Ward identities or, equivalently, the more formal methods of [34].

For $m = 1, 2, 3$ we can only have $r = 0$, and the partition function is completely determined by matching the $\mathcal{O}(\epsilon^{-2m/3})$ terms. Doing so, we easily find

$$\begin{aligned} Z_8^{(2)}(\Omega) &= 4\,\chi_{10}^{-1/3}\psi_4\,, \\ Z_{16}^{(2)}(\Omega) &= 16\,\chi_{10}^{-2/3}\psi_4^2\,, \\ Z_{24}^{(2)}(\Omega) &= \chi_{10}^{-1}\left(\frac{328}{9}\psi_4^3 + \frac{992}{9}\psi_6^2 - 7626\,\chi_{12}\right), \end{aligned} \tag{3.9}$$

with the expression in the last line being the same as that in [29, 30]. As a consistency check, one may compute the $\mathcal{O}(\epsilon^{2-2m/3})$ terms in (3.7) and (3.8) and verify that they match as well.

For $m = 4, 5$ we can have $r = 0, 1$, and determining the partition function requires that we match the order $\mathcal{O}(\epsilon^{-2m/3})$ and $\mathcal{O}(\epsilon^{2-2m/3})$ terms. To see how this matching works, let us examine the case $m = 4$ in detail. For this case, the ansatz (3.7) takes the form

$$Z_{32}^{(2)}(\Omega) = \chi_{10}^{-4/3}\left(A\psi_4^4 + B\psi_4\psi_6^2 + C\psi_4\chi_{12} + D\chi_{10}\psi_6\right), \tag{3.10}$$

where A, B, C, D are coefficients to be determined. Inserting the expansions (3.5) and (3.6) of the Siegel modular forms, noting that the leading $\epsilon^{-8/3}$ term depends only on the combinations $E_{4,i}^3/\Delta_i$ and $E_{6,i}^2/\Delta_i$, and using the identities

$$j(\tau) = \frac{E_4^3(q)}{\Delta(q)} = 1728 + \frac{E_6^2(q)}{\Delta(q)}\,, \tag{3.11}$$

we can express the result as follows

$$\begin{aligned} Z_{32}^{(2)}(\Omega) = &\frac{(j_1 j_2)^{1/3}}{\epsilon^{8/3}}\left(\frac{4A+B}{1024}j_1 j_2 - \frac{27B}{16}(j_1+j_2) + 12(243B+2C)\right) \\ &+\frac{(q_1 q_2)^{-1/3}}{\epsilon^{2/3}}\left(\frac{4A+B}{9216}\frac{1}{q_1 q_2} + \frac{124A-23B}{1152}\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\right. \\ &\left.+\frac{3844A+9277B+96C+9D}{144} + \mathcal{O}(q_i)\right) + \mathcal{O}(\epsilon^{4/3}), \end{aligned} \tag{3.12}$$

where $j_i \equiv j(\tau_i)$. Turning to the factorized expression (3.8), the $\mathcal{O}(\epsilon^{-8/3})$ and $\mathcal{O}(\epsilon^{-2/3})$ contributions are due to the torus partition function and to the one-point function of the

Virasoro descendant ϕ at level two, respectively. For this descendant field, the one-point function equals $Z^{(1)}_{32}(\phi,\tau) = q\frac{d}{dq}Z^{(1)}_{32}(\tau)$ [35] while the Zamolodchikov metric is $\mathcal{G}_{\phi\phi} = \frac{c}{2} = 4m = 16$. Hence, the relevant terms in (3.8) read

$$Z^{(2)}_{32}(\Omega) = \epsilon^{-8/3}\left(Z^{(1)}_{32}(\tau_1)Z^{(1)}_{32}(\tau_2) + \frac{1}{16}\epsilon^2\, q_1\frac{dZ^{(1)}_{32}(\tau_1)}{dq_1}q_2\frac{dZ^{(1)}_{32}(\tau_2)}{dq_2} + \mathcal{O}(\epsilon^4)\right)\,. \tag{3.13}$$

Inserting the explicit expression (2.10) for $Z^{(1)}_{32}(\tau)$ and expanding the second term in the q_i, we obtain

$$\begin{aligned} Z^{(2)}_{32}(\Omega) &= \frac{(j_1j_2)^{1/3}}{\epsilon^{8/3}}\left(j_1j_2 - 992(j_1+j_2) + 984064\right) \\ &\quad + \frac{(q_1q_2)^{-1/3}}{\epsilon^{2/3}}\left(\frac{1}{9\,q_1q_2} + \mathcal{O}(q_i)\right) + \mathcal{O}(\epsilon^{4/3})\,. \end{aligned} \tag{3.14}$$

In order for the $\mathcal{O}(\epsilon^{-8/3})$ terms in (3.12) and (3.14) to match for all values of q_1 and q_2, the coefficients of the various powers of j_1 and j_2 in the two expressions should be equal. This fixes the coefficients A, B, C to the values $A = \frac{2944}{27}$, $B = \frac{15872}{27}$ and $C = -\frac{91264}{3}$. Inserting these values in the $\mathcal{O}(\epsilon^{-2/3})$ terms in (3.12) we verify the remarkable fact that the terms of order $q_i^{-4/3}$ automatically match, while the requirement that the terms of order $(q_1q_2)^{-1/3}$ match as well fixes $D = -\frac{984064}{3}$. Therefore, our final result for the genus two partition function reads

$$Z^{(2)}_{32}(\Omega) = \chi_{10}^{-4/3}\left(\frac{2944}{27}\psi_4^4 + \frac{15872}{27}\psi_4\psi_6^2 - \frac{91264}{3}\psi_4\chi_{12} - \frac{984064}{3}\chi_{10}\psi_6\right)\,. \tag{3.15}$$

Repeating the same steps for $m = 5$, we find that the corresponding genus two partition function is given by

$$Z^{(2)}_{40}(\Omega) = \chi_{10}^{-5/3}\left(\frac{7808}{27}\psi_4^5 + \frac{79360}{27}\psi_4^2\psi_6^2 - \frac{302560}{3}\psi_4^2\chi_{12} - \frac{9840640}{3}\chi_{10}\psi_4\psi_6\right)\,. \tag{3.16}$$

The procedure can be extended to higher values of m, but becomes increasingly cumbersome since one needs higher-order terms in the ϵ- and q_i-expansions of Siegel modular forms in eqs. (3.5) and (3.6). For example, for $m = 6$ the expansions given here allow us to determine the polynomials Ψ_{24} and Ψ_{14}, in precise agreement with the results of [30], but leave the coefficient in $\Psi_4 \sim \psi_4$ undetermined. For such values of m, the method proposed in [10] is perhaps more appropriate.

To summarize this section, we have applied the method of sewing tori to compute the genus two partition functions of the extremal CFTs up to $c = 40$, providing thus a non-trivial consistency check of these theories. It would be interesting to verify the above expressions by considering the alternative degeneration limit to a single torus with the aid of the formulas in [32]. It is also important to try to construct genus two partition functions for larger values of c, where it is not known whether extremal CFTs actually exist.

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4. Discussion

In this paper, we contemplated the possibility that the conjectured duality of [10] between three-dimensional AdS gravity and extremal holomorphic CFTs at central charges that are multiples of 24 may actually apply to the more general case of central charges that are multiples of 8. Although for these cases holomorphic factorization is not possible, one can impose holomorphic factorization up to a phase and follow the reasoning of [10] to uniquely determine the partition functions of the CFTs with the required properties. Here, we explicitly computed the genus one partition functions of these theories up to $c = 88$, we gave general expressions for determining the degeneracies of states, and we also calculated the genus two partition functions of the theories up to $c = 40$.

Certainly, there is a number of open issues with the CFTs under consideration and with their conjectured relation to three-dimensional gravity. Regarding the CFTs, the fact that the existence and uniqueness of of the $c \geqslant 48$ theories has not been established requires performing further consistency checks that go beyond modular invariance; in fact, there are indications [36] that the theories with $c = 24k$ and $k \geqslant 42$ may be inconsistent. Regarding the proposed duality, we are lacking a solid argument in favor of holomorphic factorization and a rationale for including the resulting non-geometric configurations (e.g. a black hole in the holomorphic sector and AdS space in the antiholomorphic sector [10]) in the gravity path integral. Moreover, as the agreement with the semiclassical entropy is by no means compelling evidence for the duality, additional arguments in support of it are needed. We hope that further developments will shed light on these issues.

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